NORTH-HOLLAND

Hook Immanantal Inequalities for Laplacians of Trees

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ABSTRACT

For an irreducible character χ_λ of the symmetric group S_n, indexed by the partition λ, the immanant function d_λ, acting on an $n \times n$ matrix $A = (a_{ij})$, is defined as $d_\lambda(A) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$. The associated normalized immanant $\bar{d}_\lambda$ is defined as $\bar{d}_\lambda = d_\lambda / \chi_\lambda(\text{identity})$ where identity is the identity permutation. P. Heyfron has shown that for the partitions $(k, 1^{n-k})$, the normalized immanant $\bar{d}_k$ satisfies

$$\det A = \bar{d}_1(A) \leqslant \bar{d}_2(A) \leqslant \cdots \leqslant \bar{d}_n(A) = \operatorname{per} A \tag{1}$$

for all positive semidefinite Hermitian matrices A. When A is restricted to the Laplacian matrices of graphs, improvements on the inequalities above may be expected. Indeed, in a recent survey paper, R. Merris conjectured that

$$\bar{d}_{n-1}(A) \leqslant \frac{n-2}{n-1} \bar{d}_n(A) \tag{2}$$

whenever A is the Laplacian matrix of a tree. In this note, we establish a refinement for the family of inequalities in (1) when A is the Laplacian matrix of a tree, that includes (2) as a special case. These inequalities are sharp and equality holds if and only if A is the Laplacian matrix of the star. This is proved via the inequalities $\bar{d}_k(A) - \bar{d}_{k-1}(A) \leqslant \bar{d}_{k+1}(A) - \bar{d}_k(A)$ for $k = 2, 3, \ldots, n-1$, where A is the Laplacian matrix of a tree. © Elsevier Science Inc., 1997

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1. INTRODUCTION

Let χ_λ be an irreducible character of S_n, indexed by a partition λ. The immanant function d_λ associated with a character χ_λ acting on an $n \times n$ matrix A is defined as

$$d_\lambda(A) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If $\lambda = (n)$, then $\chi_{(n)}$ is the trivial character and $d_{(n)}$ is just the permanent function. For $\lambda = (1^n)$, $\chi_{(1^n)}$ is the alternating character and $d_{(1^n)}$ is the determinant function. We define the normalized immanant function $\bar{d}_\lambda$ by

$$\bar{d}_\lambda(A) = \frac{d_\lambda(A)}{d_\lambda(I_n)},$$

where I_n is the $n \times n$ identity matrix. Observe that $d_\lambda(I_n) = \chi_\lambda(\text{identity})$ is the degree of the irreducible representation whose character is χ_λ.

In 1918, I. Schur proved that

$$\det A \leqslant \bar{d}_\lambda(A)$$

for all irreducible characters χ_λ of S_n and all positive semidefinite Hermitian matrices A. There has been much work (see [6–9, 13–16]) on the establishment of the permanental analogue of Schur's inequality. This is the so-called "permanent-on-top" conjecture that asserts the inequality

$$\bar{d}_\lambda(A) \leqslant \operatorname{per} A$$

for all irreducible characters χ_λ of S_n and all positive semidefinite Hermitian matrices A. In this connection, P. Heyfron proved that:

THEOREM 1.1 [7, Theorem 1]. *The single-hook immanants* $d_k = d_{(k,1^{n-k})}$ *are ordered as*

$$\det A = \bar{d}_1(A) \leqslant \bar{d}_2(A) \leqslant \cdots \leqslant \bar{d}_{n-1}(A) \leqslant \bar{d}_n(A) = \operatorname{per} A \qquad (3)$$

for all positive semidefinite Hermitian matrices A.

The restriction of immanants to the Laplacian matrices of graphs is interesting, as it involves the symbiotic interplay of algebra and graph theory (see for instance [4, 5, 12]). For example, one may expect refinements to some of the inequalities in (3) when A is restricted to the Laplacian matrices of graphs or trees. Indeed, in his survey paper [10, Conjecture 6.5], R. Merris conjectured that if $A = L(T)$ is the Laplacian matrix of a tree T then the inequality

$$\bar{d}_{n-1}(A) \leqslant \operatorname{per} A$$

in (3) may be improved to

$$\bar{d}_{n-1}(L(T)) \leqslant \frac{n-2}{n-1} \operatorname{per} L(T). \tag{4}$$

In this work, we establish a refinement of the entire family of inequalities in (3) that includes (4) as a special case when A is the Laplacian matrix of a tree. Specifically, we claim

THEOREM 1.2. *For $k = 2, 3, \ldots, n$,*

$$\bar{d}_{k-1}(L(T)) \leqslant \frac{k-2}{k-1} \bar{d}_k(L(T)) \tag{5}$$

for all Laplacian matrices $L(T)$, where T is a tree with n vertices. Moreover, for $k > 2$, equality holds if and only if T is a star.

We introduce the definitions and notation in Section 2. Using elementary methods, we prove (5) for $k = 2, 3, n - 1$ and n in Section 3 and Section 4. A unified approach for all values of k comes from the following:

THEOREM 1.3. *Let T be a tree with n vertices. For $k = 2, 3, \ldots, n - 1$,*

$$\bar{d}_k(L(T)) - \bar{d}_{k-1}(L(T)) \leqslant \bar{d}_{k+1}(L(T)) - \bar{d}_k(L(T)). \tag{6}$$

This is proved in Section 6 using ideas from [4, 18]. These ideas are developed in Section 5.

2. IMMANANTS OF TREES

Let T be a tree on n vertices with vertex set $V(T) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(T)$. The Laplacian matrix, $L(T) = (l_{ij})$ is defined by

$$l_{ij} = \begin{cases} \deg_T(v_i) & \text{if } i = j, \\ -1 & \text{if } \{v_i, v_j\} \in E(T), \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg_T(v)$ denotes the degree of the vertex v in T. We may drop the subscript T if it is clear from the context which tree we are referring to. Since T has no cycles, $\prod_{i=1}^{n} l_{i\sigma(i)} = 0$, if, in cycle notation, σ has a cycle of length 3 or more. For permutations σ containing cycles of length at most 2, $\prod_{i=1}^{n} l_{i\sigma(i)} = 0$ if, for some $i \neq \sigma(i)$, $\{v_i, v_{\sigma(i)}\} \notin E(T)$. This shows that in the calculation of $d_\lambda(L(T))$, many of the terms are zero.

A j-matching $M \subseteq E(T)$ is a subset of j independent edges in T. We use the notation $v \in M$ to mean that the vertex v is incident with some edge in M, and $v \notin M$ will mean that the vertex v is not incident to any edge in M. For $j = 0, 1, \dots, \lfloor n/2 \rfloor$, the *weighted j-matching number* of T is defined to be

$$m_T(j) = \sum_M \prod_{v \notin M} \deg_T(v),$$

where the sum is taken over all possible j-matchings M of the tree T. If all vertices are incident to edges in M, we consider the empty product to be 1 and thus $m_T(j) = 1$. Note that $m_T(j) \geqslant 0$ and $m_T(0) = \prod \deg_T(v)$. Denote $\chi_\lambda(j) = \chi_\lambda(\sigma)$ where σ has cycle type $(2^j, 1^{n-2j})$. It is straightforward to check that

$$d_\lambda(L(T)) = \sum_{j=0}^{\lfloor n/2 \rfloor} \chi_\lambda(j) m_T(j). \tag{7}$$

Using this interpretation, it was shown in [4] that the path and the star yield the largest and smallest immanant values respectively for all irreducible characters of S_n. That is,

$$d_\lambda(L(\text{star})) \leqslant d_\lambda(L(\text{tree})) \leqslant d_\lambda(L(\text{path})), \tag{8}$$

where the inequalities are strict whenever $\lambda \neq (1^n), (2, 1^{n-2})$ and the tree is neither the star nor the path.

The expression in (7) provides an easy proof that the permanent-on-top conjecture holds for Laplacians of trees.

THEOREM 2.1. *Let T be a tree. Then*

$$\bar{d}_\lambda(L(T)) \leqslant \operatorname{per}(L(T)).$$

Proof. Recall that $|\chi_\lambda(\sigma)| \leqslant \chi_\lambda(0)$ for any permutation σ. Then

$$\begin{aligned}\bar{d}_\lambda(L(T)) &= \sum_{j \geqslant 0} \frac{\chi_\lambda(j)}{\chi_\lambda(0)} m_T(j) \\ &\leqslant \sum_{j \geqslant 0} \frac{|\chi_\lambda(j)|}{|\chi_\lambda(0)|} m_T(j) \leqslant \sum_{j \geqslant 0} m_T(j) = \operatorname{per} L(T). \qquad \blacksquare\end{aligned}$$

In the same manner, the permanent-on-top conjecture for Laplacian matrices $L(G)$ of bipartite graphs G can be proved, since bipartite graphs have no odd cycles and so all the terms $\prod_{i=1}^n L(G)_{i\sigma(i)}$ are nonnegative.

Denote the star with n vertices by s_n. The matching numbers of s_n where $n \geqslant 2$ are

$$m_{s_n}(0) = m_{s_n}(1) = n - 1 \quad \text{and} \quad m_{s_n}(j) = 0 \quad \text{for } j > 1.$$

Using these and

$$\chi_k(0) = \binom{n-1}{k-1} \quad \text{and} \quad \chi_k(1) = \binom{n-3}{k-3} - \binom{n-3}{k-1},$$

it is not difficult to obtain

$$\bar{d}_k(L(s_n)) = 2(k-1) \tag{9}$$

for $k = 1, 2, \dots, n$. With this formula, we see that

$$\bar{d}_{k-1}(s_n) = \frac{k-2}{k-1}\bar{d}_k(s_n). \tag{10}$$

This shows that Theorem 1.2 holds for stars.

Though Theorem 1.2 may be resolved in the affirmative using the inequality in Theorem 1.3, we shall single out the cases $k = 2, 3, n-1$ and n. The chief reason is that many interesting properties of the matching numbers and topological features of trees are uncovered in the process.

3. MERRIS'S CONJECTURE AND THE CASE $k = n - 1$

Consider the fraction $m_T(1)/m_T(0)$. This fraction was first used in [2] in the computation of lower bounds for per $L(G)$ where G is a bipartite graph. We state the result but restrict ourselves to trees. This admits a simple proof which we present here.

LEMMA 3.1 [2, Theorem 3.2]. *Let T be a tree of order n. Then*

$$\frac{m_T(1)}{m_T(0)} \geqslant 1. \tag{11}$$

Proof. Let Δ be the maximum degree of vertices in T. Denote the set of edges incident to a leaf of T by $\mathrm{EL}(T)$. The size of this set must be at least Δ. Hence

$$\begin{aligned}
\frac{m_T(1)}{m_T(0)} &= \sum_{\{u,v\}\in E(T)} \frac{1}{\deg(u)\deg(v)} \\
&\geqslant \sum_{\{u,v\}\in \mathrm{EL}(T)} \frac{1}{\deg(u)\deg(v)} \\
&\geqslant \sum_{\{u,v\}\in \mathrm{EL}(T)} \frac{1}{\Delta} \\
&\geqslant \Delta\frac{1}{\Delta} = 1.
\end{aligned}$$

■

We can now prove Merris's conjecture.

THEOREM 3.2. *Let T be a tree with n vertices and $n \geqslant 3$. Then*

$$\bar{d}_{n-1}(L(T)) \leqslant \frac{n-2}{n-1} \operatorname{per} L(T)$$

with equality when and only when T is a star.

Proof. It is well known that $\chi_{n-1}(j) = \chi_{(n-1,1)}(j) = n - 2j - 1$ (see [17, Example 2.3.8]). Thus

$$\begin{aligned} d_{n-1}(L(T)) &= \sum_j \chi_{n-1}(j) m_T(j) \\ &= \sum_j (n - 2j - 1) m_T(j) \\ &= \sum_j (n-2) m_T(j) + \sum_j (1 - 2j) m_T(j) \\ &= (n-2) \operatorname{per} L(T) + m_T(0) - m_T(1) - 3m_T(2) - \cdots \\ &\quad - \left(2\left\lfloor \frac{n}{2} \right\rfloor - 1\right) m_T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \\ &\leqslant (n-2) \operatorname{per} L(T), \end{aligned}$$

since $m_T(j) \geqslant 0$ for all $j \geqslant 0$ and $m_T(0) \leqslant m_T(1)$ by (11). Thus

$$\bar{d}_{n-1}(L(T)) \leqslant \frac{n-2}{n-1} \operatorname{per} L(T).$$

If equality holds, we must have $m_T(j) = 0$ for $j > 1$. This means that the tree T has diameter at most 2, which forces T to be a star. Hence, with (10), we see that equality holds if and only if T is the star. ∎

We can also use (11) to prove (5) for $k = n - 1$.

THEOREM 3.3. *Let T be a tree on n vertices, $n \geqslant 3$. Then*

$$\bar{d}_{n-2}(L(T)) \leqslant \frac{n-3}{n-2} \bar{d}_{n-1}(L(T))$$

with equality if and only if T is a star.

Proof. By using $\chi_k(0) = \binom{n-1}{k-1}$, the inequality becomes

$$2d_{n-2}(L(T)) \leqslant (n-3)d_{n-1}(L(T)).$$

We observe that

$$\chi_{n-1}(j) = n - 2j - 1,$$

$$\chi_{n-2}(j) = \tfrac{1}{2}(n-2j-1)(n-2j-2) - j$$

for $j = 0, 1, \ldots, \lfloor n/2 \rfloor$. So

$$\begin{aligned}
&(n-3)d_{n-1}(L(T)) - 2d_{n-2}(L(T)) \\
&= (n-3)\sum_{j=0}^{\lfloor n/2 \rfloor} \chi_{n-1}(j) m_T(j) - 2\sum_{j=0}^{\lfloor n/2 \rfloor} \chi_{n-2}(j) m_T(j) \\
&= \sum_{j=0}^{\lfloor n/2 \rfloor} [(n-2j)(2j-1)+1] m_T(j) \\
&= -(n-1)m_T(0) + (n-1)m_T(1) \\
&\quad + \sum_{j=2}^{\lfloor n/2 \rfloor} [(n-2j)(2j-1)+1] m_T(j) \\
&\geqslant 0
\end{aligned}$$

by (11). As before, equality implies $m_T(j) = 0$ for all $j \geqslant 2$. Therefore, T has to be a star, and by (10),

$$\bar{d}_{n-2}(L(s_n)) = \frac{n-3}{n-2}\bar{d}_{n-1}(L(s_n)). \qquad \blacksquare$$

The lack of a simple formula for $\chi_k(j)$ in general prevents one from extending this method to prove (5) for other cases.

4. THE CASE $k = 2, 3$

It is well known that for any graph G,

$$d_1(L(G)) = \det L(G) = 0.$$

This shows immediately that (5) holds for $k = 2$, since both sides are zero. For $\lambda = (2, 1^{n-2})$, it can be shown that

THEOREM 4.1 [11, Theorem 2]. *Let G be a graph with three or more vertices. Then*

$$d_2(L(G)) = 2\tau(G)|E(G)|,$$

where $\tau(G)$ is the number of spanning trees in G.

Proof. From [13, Equation (29)],

$$d_2(A) = \sum_i a_{ii} \det A(i) - \det A,$$

where $A(i)$ is the principal submatrix of A obtained by deleting the ith row and ith column. When $A = L(G)$, $\det A(i) = \tau(G)$ by the matrix tree theorem [1, Theorem 12.4]. Hence

$$d_2(L(G)) = \sum_i \deg(v_i)\, \tau(G) = 2\tau(G)|E(G)|. \qquad \blacksquare$$

If G is a tree with n vertices, the formula becomes $d_2(L(G)) = 2(n-1)$. It is difficult to find similar formulas for other immanants. However, when we restrict ourselves to trees, we can still get a graph-theoretic interpretation for $d_3(L(T))$.

LEMMA 4.2. *Let T be a tree with vertex set $V(T) = \{v_1, v_2, \dots, v_n\}$. Fix a vertex v_i of T. Then*

$$\sum_j \deg(v_j)\, d(v_i, v_j) = 2\sum_j d(v_i, v_j) - (n-1),$$

where $d(v_i, v_j)$ is the distance between v_i and v_j in the tree. Here $d(v_i, v_j)$ is the number of edges along the unique path joining the vertices v_i and v_j.

Proof. Using a combinatorial argument, we prove the equivalent statement

$$\sum_j d(v_i, v_j) = \sum_j \left[\deg(v_j) - 1\right] d(v_i, v_j) + n - 1.$$

We root T at v_i. Any vertex v_j is connected to v_i by a unique path. Let v_k be the vertex in the path closest to v_j. Then

$$d(v_i, v_j) = d(v_i, v_k) + 1.$$

Call v_j a child of v_k. Denote the set of children of v_k by $C(v_k)$. Then

$$\sum_{v_j \in C(v_k)} d(v_i, v_j) = \begin{cases} \left[\deg(v_k) - 1\right] d(v_i, v_k) + \deg(v_k) - 1 & \text{if } k \neq i, \\ \deg(v_i) & \text{if } k = i. \end{cases}$$

Now, when we sum over all v_k, we get

$$\begin{aligned} \sum_j d(v_i, v_j) &= \sum_k \sum_{v_j \in C(v_k)} d(v_i, v_j) \\ &= \sum_{k \neq i} \left[\deg(v_k) - 1\right] d(v_i, v_k) + \sum_{k \neq i} \left[\deg(v_k) - 1\right] + \deg(v_i) \\ &= \sum_k \left[\deg(v_k) - 1\right] d(v_i, v_k) + n - 1. \end{aligned}$$ ∎

THEOREM 4.3. *Let T be a tree on n vertices. Then*

$$d_3(L(T)) = 4 \sum_{i=1}^{n} \sum_{j>i} \left[d(v_i, v_j) - 1\right]. \tag{12}$$

Proof. From [13, Equation 29],

$$d_3(A) = \sum_{i=1}^{n} \sum_{j>i} (a_{ii}a_{jj} + a_{ij}a_{ji}) \det A(i,j) - d_2(A),$$

where $A(i,j)$ is the principal submatrix of A obtained by deleting the ith and jth rows and columns. When $A = L(T)$, from Theorem 4.1, $d_2(L(T)) = 2(n-1)$. By the all-minors matrix-tree theorem in [3], $\det L(T)(i,j)$ is the number of spanning forests rooted at v_i and v_j. In this case, it is equal to $d(v_i, v_j)$. Also

$$a_{ij}a_{ji} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E(T), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, applying Lemma 4.2 to the expression for $d_3(\cdot)$ above, we have

$$\begin{aligned} d_3(L(T)) &= \frac{1}{2} \sum_{i=1}^{n} \deg(v_i) \sum_{j=1}^{n} \deg(v_j)\, d(v_i, v_j) \\ &\quad + \sum_{\{v_i, v_j\} \in E(T)} d(v_i, v_j) - 2(n-1) \\ &= \frac{1}{2} \sum_{i=1}^{n} \deg(v_i) \left(2 \sum_{j=1}^{n} d(v_i, v_j) - (n-1) \right) - (n-1) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} \deg(v_i)\, d(v_i, v_j) - n(n-1) \qquad (13) \\ &= \sum_{j=1}^{n} \left(2 \sum_{i=1}^{n} d(v_i, v_j) - (n-1) \right) - n(n-1) \\ &= 4 \sum_{i=1}^{n} \sum_{j>i} d(v_i, v_j) - 2n(n-1) \\ &= 4 \sum_{i=1}^{n} \sum_{j>i} \left[d(v_i, v_j) - 1 \right]. \end{aligned}$$

■

With these formulas for $d_2(L(T))$ and $d_3(L(T))$, we can prove (5). Using

$$d_k(I_n) = \binom{n-1}{k-1},$$

(5) is equivalent to

$$(n-2)d_2(L(T)) \leqslant d_3(L(T))$$

when $k = 3$. From Theorem 4.3,

$$\begin{aligned} d_3(L(T)) &= 4\sum_{i=1}^{n}\sum_{j>i}\left[d(v_i, v_j) - 1\right] \\ &\geqslant 4\left[\binom{n}{2} - (n-1)\right] \\ &= 2(n-1)(n-2) \\ &= (n-2)d_2(L(T)). \end{aligned}$$

The inequality in the second line is obtained by observing that $n - 1$ of the terms in the sum are 0 and all the other terms are at least 1. Note that equality holds when

$$\sum_{i=1}^{n}\sum_{j>i} d(v_i, v_j) = 2\binom{n}{2} - (n-1),$$

and this occurs only when T is the star s_n. Thus we have proved:

THEOREM 4.4. *Let T be a tree with $n \geqslant 3$ vertices. Then*

$$\bar{d}_2(L(T)) \leqslant \tfrac{1}{2}\bar{d}_3(L(T)),$$

and equality holds if and only if T is a star.

5. RECURRENCE RELATIONS

We begin this section by showing how Theorem 1.3 implies (5). Suppose the inequality (6) in Theorem 1.3 is true. In particular, we would have

$$\bar{d}_2(L(T)) - \bar{d}_1(L(T)) \leqslant \bar{d}_3(L(T)) - \bar{d}_2(L(T))$$

$$\Rightarrow \quad \bar{d}_2(L(T)) \leqslant \tfrac{1}{2}\bar{d}_3(L(T)),$$

since $\bar{d}_1(L(T)) = \det L(T) = 0$. This is a special case of (5). Now, assume that

$$\bar{d}_{k-1}(L(T)) \leqslant \frac{k-2}{k-1}\bar{d}_k(L(T)).$$

We show that

$$\bar{d}_k(L(T)) \leqslant \frac{k-1}{k}\bar{d}_{k+1}(L(T)).$$

From Theorem 1.3,

$$\bar{d}_k(L(T)) - \bar{d}_{k-1}(L(T)) \leqslant \bar{d}_{k+1}(L(T)) - \bar{d}_k(L(T))$$

$$\Rightarrow \quad \bar{d}_k(L(T)) - \frac{k-2}{k-1}\bar{d}_k(L(T)) \leqslant \bar{d}_{k+1}(L(T)) - \bar{d}_k(L(T))$$

$$\Rightarrow \quad \bar{d}_k(L(T)) \leqslant \frac{k-1}{k}\bar{d}_{k+1}(L(T)).$$

This shows that (5) follows from Theorem 1.3 by induction. For the rest of this paper, we work towards a proof of Theorem 1.3. We make use of recurrence relations between normalized hook characters and matching numbers.

Let $\chi_k^n(j) = \chi_{(k,1^{n-k})}(j)$ and $\bar{\chi}_k^n(j) = \chi_k^n(j)/\chi_k^n(0)$, where $\chi_\lambda(j)$ is the value of χ_λ on permutations of cycle type $(2^j, 1^{n-2j})$. We first state a special case of the relation.

LEMMA 5.1. *When n is even,*

$$\bar{\chi}_{k+1}^{n}\left(\frac{n}{2}\right) - \bar{\chi}_{k}^{n}\left(\frac{n}{2}\right) = \frac{n}{n-1}\bar{\chi}_{k}^{n-1}\left(\frac{n}{2}-1\right).$$

We give an outline of the proof. By the Murnaghan-Nakayama rule, it can be shown that

$$\chi_{k}^{n}\left(\frac{n}{2}\right) = (-1)^{\lfloor (n-k+1)/2 \rfloor}\begin{pmatrix} \frac{n}{2}-1 \\ \left\lfloor \frac{n-k}{2} \right\rfloor \end{pmatrix}.$$

By splitting into two cases according to the parity of k, the formula follows easily, Note that when k is even, both sides are equal to zero.

Now, define $\chi_{k}^{n-1}(-1) = 0$. The general form of the relation is:

LEMMA 5.2. *For $0 < k < n$ and $0 \leqslant j \leqslant \lfloor n/2 \rfloor$,*

$$\bar{\chi}_{k+1}^{n}(j) - \bar{\chi}_{k}^{n}(j) = \frac{2j}{n-1}\bar{\chi}_{k}^{n-1}(j-1).$$

Proof. We induce on n. Note that when $j = 0$, both sides are zero, since $\bar{\chi}_{k+1}^{n}(0) = \bar{\chi}_{k}^{n}(0) = 1$. So we can assume that $j > 0$ and $j \neq n/2$ if n is even in view of Lemma 5.1. When $n = 3$,

$$\bar{\chi}_{3}^{3}(1) - \bar{\chi}_{2}^{3}(1) = 1 = \bar{\chi}_{2}^{2}(0) \quad \text{and} \quad \bar{\chi}_{2}^{3}(1) - \bar{\chi}_{1}^{3}(1) = 1 = \bar{\chi}_{1}^{2}(0),$$

agreeing with the formula.

For $n \geqslant 4$, since $2j < n$, we use the Murnaghan-Nakayama rule to get

$$\chi_{k+1}^{n}(j) = \chi_{k+1}^{n-1}(j) + \chi_{k}^{n-1}(j). \tag{14}$$

Then

$$
\begin{aligned}
\bar{\chi}^{n}_{k+1}(j) - \bar{\chi}^{n}_{k}(j) &= \frac{\chi^{n-1}_{k+1}(j) + \chi^{n-1}_{k}(j)}{\binom{n-1}{k}} - \frac{\chi^{n-1}_{k}(j) + \chi^{n-1}_{k-1}(j)}{\binom{n-1}{k-1}} \\
&= \frac{\binom{n-2}{k}}{\binom{n-1}{k}} \bar{\chi}^{n-1}_{k+1}(j) - \frac{\binom{n-2}{k-1}}{\binom{n-1}{k-1}} \bar{\chi}^{n-1}_{k}(j) \\
&\quad + \frac{\binom{n-2}{k-1}}{\binom{n-1}{k}} \bar{\chi}^{n-1}_{k}(j) - \frac{\binom{n-2}{k-2}}{\binom{n-1}{k-1}} \bar{\chi}^{n-1}_{k-1}(j) \\
&= \frac{n-k-1}{n-1} \bar{\chi}^{n-1}_{k+1}(j) - \frac{n-k}{n-1} \bar{\chi}^{n-1}_{k}(j) \\
&\quad + \frac{k}{n-1} \bar{\chi}^{n-1}_{k}(j) - \frac{k-1}{n-1} \bar{\chi}^{n-1}_{k-1}(j) \\
&= \frac{n-k-1}{n-1} \bar{\chi}^{n-1}_{k+1}(j) - \frac{n-k-1}{n-1} \bar{\chi}^{n-1}_{k}(j) \\
&\quad + \frac{k-1}{n-1} \bar{\chi}^{n-1}_{k}(j) - \frac{k-1}{n-1} \bar{\chi}^{n-1}_{k-1}(j) \\
&= \frac{n-k-1}{n-1} \frac{2j}{n-2} \bar{\chi}^{n-2}_{k}(j-1) \\
&\quad + \frac{k-1}{n-1} \frac{2j}{n-2} \bar{\chi}^{n-2}_{k-1}(j-1) \\
&= \frac{2j}{n-1} \frac{1}{\binom{n-2}{k-1}} \left[\chi^{n-2}_{k}(j-1) + \chi^{n-2}_{k-1}(j-1) \right] \\
&= \frac{2j}{n-1} \bar{\chi}^{n-1}_{k}(j-1). \qquad \blacksquare
\end{aligned}
$$

Note that Lemma 5.1 requires a separate proof because Equation (14) is not valid when n is even and $j = n/2$. The relation in Lemma 5.2 arose from a careful study of ψ-functions in [7].

Next, we derive some recurrence relations for the matching numbers that are useful in the sequel. They may be found in [4, 18]. Let $v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_s \in V(T)$. We define

$$m_T(j; v_1, \ldots, v_r, \bar{v}_{r+1}, \ldots, \bar{v}_s) = \sum \prod_{v \notin M} \deg_T(v),$$

where the sum is taken over all j-matchings M such that $v_1, v_2, \ldots, v_r \in M$ and $v_{r+1}, \ldots, v_s \notin M$.

LEMMA 5.3. *Let T be a tree with $n - p$ vertices where $1 \leqslant p < n$. Let U be a tree on n vertices that contains T as a subtree as shown in Figure 1. Let $d = \deg_T(v_{n-p})$. For $j = 0, 1, \ldots, \lfloor n/2 \rfloor$, we have*

$$m_U(j) = m_T(j) + \frac{p}{d}\left[m_T(j; \bar{v}_{n-p}) + m_T(j-1; \bar{v}_{n-p})\right].$$

Proof.

$$\begin{aligned} m_U(j) &= m_U(j; v_{n-p}) + m_U(j; \bar{v}_{n-p}) \\ &= \sum_{i=1}^{p} m_U(j; v_{n-p+i}) + m_U(j; v_{n-p}, \bar{v}_{n-p+1}, \ldots, \bar{v}_n) + m_U(j; \bar{v}_{n-p}) \\ &= \sum_{i=1}^{p} \frac{1}{d} m_T(j-1; \bar{v}_{n-p}) + m_T(j; v_{n-p}) + \frac{d+p}{d} m_T(j; \bar{v}_{n-p}) \\ &= m_T(j) + \frac{p}{d}\left[m_T(j; \bar{v}_{n-p}) + m_T(j-1; \bar{v}_{n-p})\right]. \end{aligned}$$

■

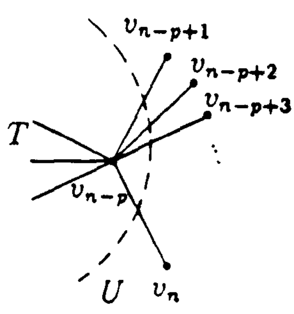

FIG. 1.

LEMMA 5.4. *Let T be a tree with $n - p$ vertices where $1 < p < n$, and T_1, T_2 be trees on n vertices containing T as a subtree as shown in Figure 2. Then, for $j = 0, 1, \ldots, \lfloor n/2 \rfloor$,*

$$m_{T_1}(j) - m_{T_2}(j) = (p - 1)[m_T(j) + m_T(j - 1)].$$

Proof. Let $d = \deg_T(v_{n-p})$. By the previous lemma, for $j = 0, 1, \ldots, \lfloor n/2 \rfloor$,

$$m_{T_2}(j) = m_T(j) + \frac{p}{d}\left[m_T(j; \bar{v}_{n-p}) + m_T(j - 1; \bar{v}_{n-p})\right].$$

Let T' be the tree obtained by attaching a vertex v_{n-p+1} to v_{n-p} of T. Again, by the previous lemma,

$$\begin{aligned} m_{T_1}(j) &= m_{T'}(j) + (p - 1)\left[m_{T'}(j; \bar{v}_{n-p+1}) + m_{T'}(j - 1; \bar{v}_{n-p+1})\right] \\ &= m_T(j) + \frac{1}{d}\left[m_T(j; \bar{v}_{n-p}) + m_T(j - 1; \bar{v}_n)\right] + (p - 1) \\ &\quad \times \left(m_T(j) + m_T(j - 1) + \frac{m_T(j; \bar{v}_{n-p})}{d} + \frac{m_T(j - 1; \bar{v}_{n-p})}{d} \right) \\ &= p m_T(j) + (p - 1) m_T(j - 1) \\ &\quad + \frac{p}{d}\left[m_T(j; \bar{v}_{n-p}) + m_T(j - 1; \bar{v}_{n-p})\right]. \end{aligned}$$

Taking the difference of the two expressions gives the desired result. ■

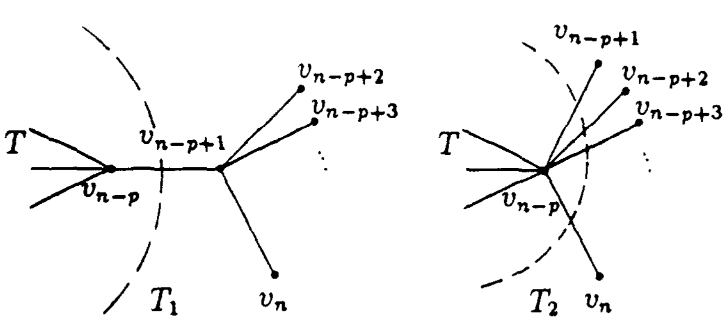

FIG. 2.

The next lemma is analogous to [4, Lemma 2; 18, Lemmas 17, 19].

LEMMA 5.5. *Let $n \geqslant 1$ be a positive integer, and let $\{a_i\}_{i \geqslant 0}$ be a sequence of real numbers for which $a_i = 0$ for all $i > \lfloor n/2 \rfloor$. We set $a_{-1} := 0$. If*

$$\sum_{j \geqslant 0} \chi_k^n(j) a_j \geqslant 0$$

for all hook characters χ_k^n, $k = 1, 2, \ldots, n$, of S_n, then

1. *$\sum_{j \geqslant 0} \chi_k^m(j) a_j \geqslant 0$ for all hook characters χ_k^m, $k = 1, 2, \ldots, m$, of S_m with $m \geqslant n$;*
2. *$\sum_{j \geqslant 0} \chi_k^m(j)(a_j + a_{j-1}) \geqslant 0$ for all hook characters χ_k^m, $k = 1, 2, \ldots, m$, of S_m with $m \geqslant n + 2$.*

Proof. 1: Let $\sigma \in S_n$. Since S_n is a subgroup of S_m, using the branching rule [17, Theorem 2.8.3], when we restrict the character χ_k^m to S_n, we get

$$\chi_k^m(\sigma) = \sum_{i=1}^{n} c_i \chi_i^n(\sigma)$$

where c_i are nonnegative integers independent of σ. Therefore,

$$\sum_{j \geqslant 0} \chi_k^m(j) a_j = \sum_{i=1}^{n} c_i \sum_{j \geqslant 0} \chi_i^n(j) a_j \geqslant 0.$$

2: Using part 1, is suffices to prove the inequality when $m = n + 2$:

$$\begin{aligned}
&\sum_{j \geqslant 0} \chi_k^{n+2}(j)(a_j + a_{j-1}) \\
&\quad = \sum_{j \geqslant 0} \chi_k^{n+2}(j) a_j + \sum_{j \geqslant 0} \chi_k^{n+2}(j+1) a_j \\
&\quad = \sum_{j \geqslant 0} \left[\chi_k^n(j) + 2\chi_{k-1}^n(j) + \chi_{k-2}^n(j) \right] a_j + \sum_{j \geqslant 0} \left[\chi_{k-2}^n(j) - \chi_k^n(j) \right] a_j \\
&\quad = \sum_{j \geqslant 0} \left[2\chi_{k-1}^n(j) + 2\chi_{k-2}^n(j) \right] a_j \\
&\quad \geqslant 0.
\end{aligned}$$

■

6. PROOF OF THEOREM 1.3

Denote the difference $\bar{d}_{k+1}(A) - \bar{d}_k(A)$ by $D_k(A)$. Then the inequality (6) in Theorem 1.3 becomes

$$D_k(L(T)) \geqslant D_{k-1}(L(T)).$$

For trees with a small number of vertices, it can be verified that Theorem 1.3 holds. For example, when $n = 3$, there is only one tree, and it can be considered as the star s_3. Using (9),

$$D_2(s_3) = \bar{d}_3(s_3) - \bar{d}_2(s_3) = 4 - 2 = 2,$$

$$D_1(s_3) = \bar{d}_2(s_3) - \bar{d}_1(s_3) = 2 - 0 = 2.$$

For $n = 4$, there are two trees, namely, the star s_4 and the path p_4. Then

$$D_3(s_4) = \bar{d}_4(s_4) - \bar{d}_3(s_4) = 6 - 4 = 2,$$

$$D_2(s_4) = \bar{d}_3(s_4) - \bar{d}_2(s_4) = 4 - 2 = 2,$$

$$D_1(s_4) = \bar{d}_2(s_4) - \bar{d}_1(s_4) = 2 - 0 = 2;$$

$$D_3(p_4) = \bar{d}_4(p_4) - \bar{d}_3(p_4) = 10 - \tfrac{16}{3} = \tfrac{14}{3},$$

$$D_2(p_4) = \bar{d}_3(p_4) - \bar{d}_2(p_4) = \tfrac{16}{3} - 2 = \tfrac{10}{3},$$

$$D_1(p_4) = \bar{d}_2(p_4) - \bar{d}_1(p_4) = 2 - 0 = 2.$$

We induce on the number of vertices to prove the result. Let T be a tree with more than four vertices. Applying the relation in Lemma 5.2, we obtain the formulas

$$D_k(L(T)) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{2(j+1)}{n-1} \bar{\chi}_k^{n-1}(j) m_T(j+1) \tag{15}$$

for $k = 1, 2, \ldots, n-1$, and

$$D_k(L(T)) - D_{k-1}(L(T)) = \sum_{j=0}^{\lfloor n/2 \rfloor - 2} \frac{4(j+1)(j+2)}{(n-1)(n-2)} \bar{\chi}_{k-1}^{n-2}(j) m_T(j+2) \tag{16}$$

for $k = 2, 3, \ldots, n-1$. Recall from Theorem 1.1 that $\bar{d}_{k+1}(A) \geqslant \bar{d}_k(A)$ if A is a positive semidefinite matrix. This implies that the sum in (15) is nonnegative.

Next, we make use of the idea behind the proof of [4, Theorem 1; 18, Theorem 25]. Given a tree T, we construct a sequence of trees $T = T_1, T_2, \ldots, T_l = s_n$ such that

$$D_k(L(T_i)) - D_{k-1}(L(T_i)) \geqslant D_k(K(T_{i+1})) - D_{k-1}(L(T_{i+1}))$$

for $i = 1, 2, \ldots, l-1$. This will show that the difference $D_k(L(T)) - D_{k-1}(L(T))$ is smallest for the star. The trees are constructed in the following manner.

Suppose we have obtained T_i and $T_i \neq s_n$. Let $v_1, v_2, \ldots, v_m$ be the vertices of a longest path in T_i. Let $N = \{v \in V(T_i) : \{v, v_{m-1}\} \in E(T_i), v \neq v_{m-2}\}$. Note that N is not empty, since it contains v_m. Form T_{i+1} by connecting all the vertices in N to v_{m-2} instead of v_{m-1}. Clearly, after repeating the construction enough times, we get a sequence of trees beginning with T and ending with s_n.

Now, the trees T_i and T_{i+1} differ in the manner described in Lemma 5.4, with a common subtree which we denote by R_i. In other words, R_i is the subtree of T_i formed by all vertices except v_{m-1} and those in N. By Lemma 5.4,

$$m_{T_i}(j) - m_{T_{i+1}}(j) = (p-1)\left[m_{R_i}(j) + m_{R_i}(j-1)\right],$$

where $p = |N| + 1$. Using this and (16),

$$D_k(L(T_i)) - D_{k-1}(L(T_i)) - \left[D_k(L(T_{i+1})) - D_{k-1}(L(T_{i+1}))\right]$$

$$= \sum_{j=0}^{\lfloor n/2 \rfloor - 2} \frac{4(j+1)(j+2)}{(n-1)(n-2)} \bar{\chi}_{k-1}^{n-2}(j)\left[m_{T_i}(j+2) - m_{T_{i+1}}(j+2)\right]$$

$$= \frac{4(p-1)}{(n-1)(n-2)} \sum_{j=0}^{\lfloor n/2\rfloor - 2} (j+1)(j+2)\bar{\chi}_{k-1}^{n-2}(j)$$

$$\times\left[m_{R_i}(j+2) + m_{R_i}(j+1)\right]$$

$$= \frac{4(p-1)}{(n-1)(n-2)} \sum_{j=0}^{\lfloor n/2\rfloor - 2} \bar{\chi}_{k-1}^{n-2}(j)\big[(j+1)(j+2)m_{R_i}(j+2)$$

$$+j(j+1)m_{R_i}(j+1)\big]$$

$$+\frac{4(p-1)}{(n-1)(n-2)} \sum_{j=0}^{\lfloor n/2\rfloor - 2} 2(j+1)\bar{\chi}_{k-1}^{n-2}(j)m_{R_i}(j+1)$$

$$= \frac{4(p-1)}{(n-1)(n-2)\chi_{k-1}^{n-2}(0)} \sum_{j\geqslant 0} \chi_{k-1}^{n-2}(j)(a_j + a_{j-1})$$

$$+\frac{4(p-1)}{(n-1)(n-2)\chi_{k-1}^{n-2}(0)} \sum_{j\geqslant 0} \chi_{k-1}^{n-2}(j)b_j,$$

where $a_j = (j+1)(j+2)m_{R_i}(j+2)$ and $b_j = 2(j+1)m_{R_i}(j+1)$. Note that

$$a_j = 0 \quad \text{for } j > \left\lfloor \frac{n-p}{2} \right\rfloor - 2 \quad \text{and} \quad b_j = 0 \quad \text{for } j > \left\lfloor \frac{n-p}{2} \right\rfloor - 1.$$

Now by induction, for $k = 1, 2, \ldots, n-p-1$,

$$D_k(L(R_i)) - D_{k-1}(L(R_i)) \geqslant 0$$

$$\Rightarrow \sum_{j=0}^{\lfloor (n-p)/2\rfloor} \frac{4(j+1)(j+2)}{(n-p-1)(n-p-2)} \bar{\chi}_{k-1}^{n-p-2}(j)m_{R_i}(j+2) \geqslant 0$$

$$\Rightarrow \quad \sum_{j\geqslant 0} \chi_{k-1}^{n-p-2}(j)a_j \geqslant 0,$$

and from Theorem 1.1,

$$D_k(K(R_i)) = \sum_{j=0}^{\lfloor (n-p)/2 \rfloor - 1} \frac{2(j+1)}{n-p-1} \bar{\chi}_k^{n-p-1}(j) m_{R_i}(j+1) \geqslant 0$$

$$\Rightarrow \quad \sum_{j \geqslant 0} \chi_k^{n-p-1}(j) b_j \geqslant 0.$$

It follows from Lemma 5.5 that each of the sums in the calculation is nonnegative. Hence

$$D_k(L(T_i)) - D_{k-1}(L(T_i)) \geqslant D_k(L(T_{i+1})) - D_{k-1}(L(T_{i+1})).$$

For the star s_n, using (9),

$$D_k(L(s_n)) = \bar{d}_{k+1}(L(s_n)) - \bar{d}_k(L(s_n)) = 2$$

for all $k = 1, 2, \ldots, n-1$, and therefore, for any tree T,

$$D_k(L(T)) - D_{k-1}(L(T)) \geqslant D_k(L(s_n)) - D_{k-1}(L(s_n)) = 0,$$

and so

$$\begin{aligned} D_{k-1}(L(T)) &= \bar{d}_k(L(T)) - \bar{d}_{k-1}(L(T)) \\ &\leqslant \bar{d}_{k+1}(L(T)) - \bar{d}_k(L(T)) = D_k(L(T)). \end{aligned}$$

This completes the proof of Theorem 1.3.

7. EQUALITY AND THE REST

From the discussion in Section 5, we see that Theorem 1.2 is now proved except for the case of equality. Suppose for a given tree T,

$$\bar{d}_k(L(T)) = \frac{k-1}{k} \bar{d}_{k+1}(L(T))$$

for some k, $2 \leqslant k < n$. Now, as we have proved in Theorem 1.3,

$$\bar{d}_k(L(T)) - \bar{d}_{k-1}(L(T)) \leqslant \bar{d}_{k+1}(L(T)) - \bar{d}_k(L(T))$$

$$\Rightarrow \quad \bar{d}_k(L(T)) - \bar{d}_{k-1}(L(T)) \leqslant \frac{k}{k-1}\bar{d}_k(L(T)) - \bar{d}_k(L(T))$$

$$\Rightarrow \quad \frac{k-2}{k-1}\bar{d}_k(L(T)) \leqslant \bar{d}_{k-1}(L(T)).$$

But by (5),

$$\bar{d}_{k-1}(L(T)) \leqslant \frac{k-2}{k-1}\bar{d}_k(L(T)).$$

Hence,

$$\bar{d}_{k-1}(L(T)) = \frac{k-2}{k-1}\bar{d}_k(L(T)).$$

By repeating the process, we eventually get

$$\bar{d}_2(L(T)) = \tfrac{1}{2}\bar{d}_3(L(T)).$$

From Theorem 4.5, this occurs only when $T = s_n$. This completes the proof of Theorem 1.2.

The conjugate of the hook partition $(k, 1^{n-k})$ is the hook partition $(n-k+1, 1^{k-1})$. An immediate consequence of the decreasing gaps between consecutive hook immanants in Theorem 1.3 is a relation involving the sum of the normalized immanant $\bar{d}_k$ and its conjugate $\bar{d}_{n-k+1}$.

COROLLARY 7.1. *Let T be a tree with n vertices. Then for j, k such that $\lfloor n/2 \rfloor \leqslant j < k \leqslant n$,*

$$\bar{d}_j(L(T)) + \bar{d}_{n-j+1}(L(T)) \leqslant \bar{d}_k(L(T)) + \bar{d}_{n-k+1}(L(T)).$$

In particular,

$$\bar{d}_j(L(T)) + \bar{d}_{n-j+1}(L(T)) \leqslant \operatorname{per} L(T),$$

where $j = 1, 2, \ldots, n$.

Proof. For $\lfloor n/2 \rfloor \leqslant j < k \leqslant n$,

$$\begin{aligned} &\bar{d}_{n-k+1}(L(T)) - \bar{d}_{n-k+2}(L(T)) \leqslant \bar{d}_k(L(T)) - \bar{d}_{k-1}(L(T)) \\ \Rightarrow\quad &\bar{d}_{k-1}(L(T)) + \bar{d}_{n-k+2}(L(T)) \leqslant \bar{d}_k(L(T)) + \bar{d}_{n-k+1}(L(T)) \\ \Rightarrow\quad &\bar{d}_j(L(T)) + \bar{d}_{n-j+1}(L(T)) \leqslant \bar{d}_k(L(T)) + \bar{d}_{n-k+1}(L(T)). \end{aligned}$$

Note that $\bar{d}_n(L(T)) = \operatorname{per} L(T)$ and $\bar{d}_1 = \det L(T) = 0$. So when $k = n$, the inequality becomes

$$\bar{d}_j(L(T)) + \bar{d}_{n-j+1}(L(T)) \leqslant \operatorname{per} L(T). \qquad \blacksquare$$

Though it is tempting to extend the results to graphs in general, Theorem 1.2 and Theorem 1.3 do not hold for graphs. For example, consider the complete graph, K_3 on three vertices. We have $\bar{d}_1(L(K_3)) = 0$, $\bar{d}_2(L(K_3)) = 9$, and $\bar{d}_3(L(K_3)) = 12$. So

$$\bar{d}_2(L(K_3)) > \tfrac{1}{2}\bar{d}_3(L(K_3))$$

and

$$\bar{d}_2(L(K_3)) - \bar{d}_1(L(K_3)) > \bar{d}_3(L(K_3)) - \bar{d}_2(L(K_3)).$$

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